

PROVA SCRITTA DI ISTITUZIONI DI MATEMATICHE II

Scienze Ambientali

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Appello del 10 Giugno 2004

1. Calcolare la derivata della seguente funzione

$$f(x) = \sin x e^{x^2-x}.$$

2. Calcolare il seguente limite

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x^2}.$$

3. Calcolare il seguente integrale

$$\int \frac{x^2 + 2}{x^2 - 1} dx.$$

4. Determinare a e b in modo che la funzione $f : \mathbb{R} \rightarrow \mathbb{R}$ definita da

$$f(x) = \begin{cases} e^x - \cos x & \text{se } x > 0 \\ x^2 + ax + b & \text{se } x \leq 0 \end{cases}$$

sia continua e derivabile in $x = 0$.

5. Studiare la funzione

$$f(x) = x^2 - \sqrt{x^2 + 1} \quad y'' \text{ dy da}$$

e tracciarne il grafico.

6. La funzione $f : [-1, 1] \rightarrow \mathbb{R}$ definita da $f(x) = x^2 - |x|$ verifica le ipotesi del teorema di Lagrange?

$$y' = 2x - \frac{1}{2\sqrt{x^2+1}} \cdot 2x = 2x - \frac{x}{\sqrt{x^2+1}} = \frac{2x\sqrt{x^2+1} - x}{\sqrt{x^2+1}} \quad \text{A} =$$

$$= \frac{x(2\sqrt{x^2+1} - 1)}{\sqrt{x^2+1}} \geq 0 \quad \begin{cases} \sqrt{x^2+1} > 0 \quad \forall x \\ x \geq 0 \\ 2\sqrt{x^2+1} - 1 \geq 0 \Rightarrow \sqrt{x^2+1} \geq \frac{1}{2} \end{cases}$$

Donc que $\frac{1}{2} > 0$ alors d'où nous avons $x^2+1 \geq \left(\frac{1}{2}\right)^2 \Rightarrow x^2+1 - \frac{1}{4} \geq 0$
 $\Rightarrow x^2 + \frac{3}{4} \geq 0 \quad \forall x$

$$\begin{array}{c} 0 \\ \hline - \quad - \quad - \quad - \quad - \\ \hline - \quad + \quad - \quad + \quad - \\ \hline \end{array}$$

P(0;1) milieu

$$y'' = \frac{\left((2\sqrt{x^2+1} - 1) + x \cdot \frac{2 \cdot 1}{2\sqrt{x^2+1}} \cdot 2x \right) \cdot \sqrt{x^2+1} - x(2\sqrt{x^2+1} - 1) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x}{x^2+1} =$$

$$= \frac{\frac{(2\sqrt{x^2+1} - 1)(\sqrt{x^2+1}) + 2x^2 \cdot \sqrt{x^2+1} - \frac{x^2(2\sqrt{x^2+1} - 1)}{\sqrt{x^2+1}}}{x^2+1} =$$

$$= \frac{(2\sqrt{x^2+1} - 1)(x^2+1) + 2x^2\sqrt{x^2+1} - \frac{2x^2\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}}}{(x^2+1)} =$$

$$= \frac{\cancel{2x^2\sqrt{x^2+1}} + 2\sqrt{x^2+1} - x^2 - 1 + \cancel{2x^2\sqrt{x^2+1}} - \cancel{2x^2\sqrt{x^2+1}} + \cancel{x^2}}{\sqrt{x^2+1}(x^2+1)} =$$

$$= \frac{2\sqrt{x^2+1} - x^2 - 1 + 2x^2}{\sqrt{x^2+1}(x^2+1)} = \frac{2\sqrt{x^2+1} + x^2 - 1}{\sqrt{x^2+1}(x^2+1)}$$

$$\frac{2x^2\sqrt{x^2+1} + 2\sqrt{x^2+1} - 1}{\sqrt{x^2+1}(x^2+1)} = \frac{2\sqrt{x^2+1}(x^2+1) - 1}{\sqrt{x^2+1}(x^2+1)} \geq 0$$

$$2\sqrt{x^2+1}(x^2+1) - 1 \geq 0 \Rightarrow \sqrt{x^2+1} \geq \frac{1}{2(x^2+1)} \Rightarrow x^2+1 > \frac{1}{4(x^2+1)^2}$$

$$y'' = 2 - \frac{\sqrt{x^2+1} - x \frac{1}{2\sqrt{x^2+1}} \cdot 2x}{x^2+1} = 2 - \frac{\frac{(x^2+1) - x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= 2 - \frac{x^2+1 - x^2}{\sqrt{x^2+1} (x^2+1)} = 2 - \frac{1}{\sqrt{x^2+1} (x^2+1)} = \frac{2\sqrt{x^2+1} (x^2+1) - 1}{\sqrt{x^2+1} (x^2+1)} \geq 0$$

$$2\sqrt{x^2+1} (x^2+1) - 1 \geq 0 \Rightarrow \sqrt{x^2+1} \geq \frac{1}{2(x^2+1)} \Rightarrow x^2+1 \geq \frac{1}{4(x^2+1)^2}$$

$$\Rightarrow x^2+1 - \frac{1}{4(x^2+1)^2} = \frac{(x^2+1)^3 - 1}{4(x^2+1)^2} \geq 0$$

$$4(x^2+1)^3 - 1 \geq 0 \Rightarrow (x^2+1)^3 \geq \frac{1}{4} \Rightarrow x^2+1 \geq \sqrt[3]{\frac{1}{4}} \Rightarrow$$

$$\Rightarrow x^2+1 - \frac{1}{\sqrt[3]{4}} \geq 0 \Rightarrow \forall x \text{ sempre positivo}$$

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$$\Rightarrow \overbrace{\hspace{10em}}^{+}$$

Calculo os. obl.

$$\lim_{x \rightarrow +\infty} \frac{x^2 - \sqrt{x^2+1}}{x} = x - \frac{\sqrt{x^2+1}}{x} = \lim_{x \rightarrow +\infty} x - \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} = x - \frac{|x| \sqrt{1+\frac{1}{x^2}}}{x} =$$

$$= x - 1 = +\infty$$

$$\lim_{x \rightarrow -\infty} = x + 1 = +\infty$$

} $\nexists$ obl.

$$f(x) = \sin x \cdot e^{x^2-x}$$

$$f'(x) = \cos x \cdot e^{x^2-x} + \sin x \cdot e^{x^2-x} (2x-1)$$

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin^2 x}$$

$$\int \frac{x^2+2}{x^2-1} dx =$$

$$\begin{array}{r} \cancel{x^2+0x+2} \\ x^2 \quad -1 \quad 1 \\ \hline \quad \quad 0 \quad +3 \end{array}$$

$$= \int 1 + \frac{3}{x^2-1} dx = \int dx + 3 \int \frac{dx}{x^2-1} =$$

$$= x + 3 \cdot \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = x + \frac{3}{2} \log \left| \frac{x-1}{x+1} \right| + C$$

oss.

$$\int \frac{1}{x^2-1} = \int \frac{1}{(x+1)(x-1)} dx = \int \frac{A}{x+1} + \frac{B}{x-1} dx = \int \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} dx =$$

$$\int \frac{Ax - A + Bx + B}{(x+1)(x-1)} dx = \int \frac{(A+B)x - A+B}{(x+1)(x-1)} dx \quad \begin{cases} A+B=0 \\ -A+B=1 \end{cases} \Rightarrow$$

$$\Rightarrow \cancel{A=B} \quad A = -B \Rightarrow A = -\frac{1}{2}$$

$$-(-B) + B = 1 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2} \quad \Rightarrow$$

$$\Rightarrow = \int \frac{-\frac{1}{2}}{x+1} + \int \frac{\frac{1}{2}}{x-1} \Rightarrow = -\frac{1}{2} \log |x+1| + \frac{1}{2} \log |x-1| =$$

$$= \frac{1}{2} (\log |x-1| - \log |x+1|) = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right|$$